

QUASI VALUATION AND VALUATION DERIVED FROM FILTERED RING AND THEIR PROPERTIES

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Abstract

In this paper we show if R is a filtered ring then we can define a quasi valuation. And if R is some kind of filtered ring then we can define a valuation. Then we prove some properties and relations for R .

Key Words: Filtered ring, Quasi valuation ring, Valuation ring.

1 Introduction

In algebra valuation ring and filtered ring are two most important structure [5],[6],[7]. We know that filtered ring is also the most important structure since filtered ring is a base for graded ring especially associated graded ring and completion and some similar results, on the Andreadakis–Johnson filtration of the automorphism group of a free group [1], on the depth of the associated graded ring of a filtration [2],[3]. So, as these important structures, the relation between these structure is useful for finding some new structures, and if R is a discrete valuation ring then R has many properties that have many usage for example Decidability of the theory of modules over commutative valuation domains [7], Rees valuations and asymptotic primes of rational powers in Noetherian rings and lattices [6].

In this article we investigate the relation between filtered ring and valuation and quasi valuation ring. We prove that if we have filtered ring then we can find a quasi valuation on it. Continuously we show that if R be a strongly filtered then exist a valuation, Similarly we prove it for PID. At the end we explain some properties for them.

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2 Preliminaries

Definition 2.1. A filtered ring R is a ring together with a family $\{R_n\}_{n \geq 0}$ of subgroups of R satisfying in the following conditions

- i) $R_0 = R$;
- ii) $R_{n+1} \subseteq R_n$ for all $n \geq 0$;
- iii) $R_n R_m \subseteq R_{n+m}$ for all $n, m \geq 0$.

Definition 2.2. Let R be a ring together with a family $\{R_n\}_{n \geq 0}$ of subgroups of R satisfying the following conditions:

- i) $R_0 = R$;
- ii) $R_{n+1} \subseteq R_n$ for all $n \geq 0$;
- iii) $R_n R_m = R_{n+m}$ for all $n, m \geq 0$,

Then we say R has a strong filtration.

Definition 2.3. Let R be a ring and I an ideal of R . Then $R_n = I^n$ is called I -adic filtration.

Definition 2.4. A map $f : M \rightarrow N$ is called a homomorphism of filtered modules if: (i) f is R -module homomorphism and (ii) $f(M_n) \subseteq N_n$ for all $n \geq 0$.

Definition 2.5. A subring R of a field K is called a valuation ring of K if for every $\alpha \in K$, $\alpha \neq 0$, either $\alpha \in R$ or $\alpha^{-1} \in R$.

Definition 2.6. Let Δ be a totally ordered abelian group. A valuation ν on R with values in Δ is a mapping $\nu : R^* \rightarrow \Delta$ satisfying :

- i) $\nu(ab) = \nu(a) + \nu(b)$;
- ii) $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$.

Definition 2.7. Let Δ be a totally ordered abelian group. A quasi valuation ν on R with values in Δ is a mapping $\nu : R^* \rightarrow \Delta$ satisfying :

- i) $\nu(ab) \geq \nu(a) + \nu(b)$;
- ii) $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$.

Remark 2.1. R is said to be **vaulted ring**; $R_\nu = \{x \in R : \nu(x) \geq 0\}$ and $\nu^{-1}(\infty) = \{x \in R : \nu(x) = \infty\}$.

Definition 2.8. Let K be a field. A discrete valuation on K is a valuation $\nu : K^* \rightarrow \mathbb{Z}$ which is surjective.

Theorem 2.1. *If R is a UFD then R is a PID (see [2]).*

Proposition 2.1. *Any discrete valuation ring is a Euclidean domain(see[3]).*

Remark 2.2. *If R is a ring, we will denote by $Z(R)$ the set of **zero-divisors** of R and by $T(R)$ the **total ring of fractions** of R .*

Definition 2.9. *A ring R is said to be a **Manis valuation ring** (or simply a **Manis ring**) if there exist a valuation ν on its total fractions $T(R)$, such that $R = R_\nu$.*

Definition 2.10. *A ring R is said to be a **Prüfer ring** if each overring of R is integrally closed in $T(R)$.*

Definition 2.11. *A Manis ring R_ν is said to be **ν -closed** if $R_\nu/\nu^{-1}(\infty)$ is a valuation domain (see Theorem 2 of [8]).*

3 Quasi Valuation and Valuation derived from Filtered ring

Let R be a ring with unit and R a filtered ring with filtration $\{R_n\}_{n \geq 0}$.

Lemma 3.1. *Let R be a filtered ring with filtration $\{R_n\}_{n \geq 0}$. Now we define $\nu : R \rightarrow \mathbb{Z}$ such that for every $\alpha \in R$ and $\nu(\alpha) = \min \{i | \alpha \in R_i \setminus R_{i+1}\}$. Then we have $\nu(\alpha\beta) \geq \nu(\alpha) + \nu(\beta)$.*

Proof. For any $\alpha, \beta \in R$ with $\nu(\alpha) = i$ and $\nu(\beta) = j$, $\alpha\beta \in R_i R_j \subseteq R_{i+j}$. Now let $\nu(\alpha\beta) = k$ then we have $\alpha\beta \in R_k \setminus R_{k+1}$.

We show that $k \geq i + j$.

Let $k < i + j$ so we have $k + 1 \leq i + j$ hence $R_{k+1} \supset R_{i+j}$ then $\alpha\beta \in R_{i+j} \subseteq R_{k+1}$ it is contradiction. So $k \geq i + j$. Now we have $\nu(\alpha\beta) \geq \nu(\alpha) + \nu(\beta)$. □

Lemma 3.2. *Let R be a filtered ring with filtration $\{R_n\}_{n \geq 0}$. Now we define $\nu : R \rightarrow \mathbb{Z}$ such that for every $\alpha \in R$ and $\nu(\alpha) = \min \{i | \alpha \in R_i \setminus R_{i+1}\}$. Then $\nu(\alpha + \beta) \geq \min \{\nu(\alpha), \nu(\beta)\}$.*

Proof. For any $\alpha, \beta \in R$ such that $\nu(\alpha) = i$ also $\nu(\beta) = j$ and $\nu(\alpha + \beta) = h$ so we have $\alpha + \beta \in R_h \setminus R_{h+1}$. Without losing the generality, let $i < j$ so $R_j \subset R_i$ hence $\beta \in R_i$. Now if $k < i$ then $k + 1 \leq i$ and $R_i \subset R_{k+1}$ so $\alpha + \beta \in R_i \subset R_{k+1}$ it is contradiction. Hence $k \geq i$ and so we have $\nu(\alpha + \beta) \geq \min \{\nu(\alpha), \nu(\beta)\}$. □

Theorem 3.1. *Let R be a filtered ring. Then there exist a quasi valuation on R .*

Proof. Let R be a filtered ring with filtration $\{R_n\}_{n \geq 0}$. Now we define $\nu : R \rightarrow \mathbb{Z}$ such that for every $\alpha \in R$ and $\nu(\alpha) = \min \{i \mid \alpha \in R_i \setminus R_{i+1}\}$. Then

- i) By lemma(3.1) we have $v(\alpha\beta) \geq v(\alpha) + v(\beta)$.
- ii) By lemma(3.2) we have $v(\alpha + \beta) \geq \min \{v(\alpha), v(\beta)\}$. So by 2.7 R is quasi valuation ring.

□

Proposition 3.1. *Let R be a strongly filtered ring. Then there exists a valuation on R .*

Proof. By theorem (3.1) we have $v(\alpha\beta) \geq v(\alpha) + v(\beta)$ and $v(\alpha + \beta) \geq \min \{v(\alpha), v(\beta)\}$. Now we show $v(\alpha\beta) = v(\alpha) + v(\beta)$. Let $v(\alpha\beta) > v(\alpha) + v(\beta)$ so $k > i + j$ and it is contradiction. So $v(\alpha\beta) = v(\alpha) + v(\beta)$, then there is a valuation on R .

□

Corollary 3.1. *Let R be a strongly filtered ring, then R is a Euclidean domain.*

Proof. By proposition (3.1) R is a discrete valuation and so by proposition (2.1) R is a Euclidean domain.

□

Proposition 3.2. *Let P is a prime ideal of R and $\{P^n\}_{n \geq 0}$ be P -adic filtration. Then there exists a valuation on R .*

Proof. By theorem (3.1) we have $v(\alpha\beta) \geq v(\alpha) + v(\beta)$ and $v(\alpha + \beta) \geq \min \{v(\alpha), v(\beta)\}$. Now we show $v(\alpha\beta) = v(\alpha) + v(\beta)$. Let $v(\alpha\beta) > v(\alpha) + v(\beta)$ so $k > i + j$ then $\alpha\beta \in P^k \subset P^{i+j}$ and $k \geq i + j + 1$, since P is a prime ideal hence $\alpha \in P^{i+1}$ or $\beta \in P^{j+1}$ and it is contradiction. So $v(\alpha\beta) = v(\alpha) + v(\beta)$, then there is a valuation on R .

□

Proposition 3.3. *Let R be a PID then there is a valuation on R .*

Proof. By theorem (3.1) and proposition (3.2) there is a valuation on R .

□

Corollary 3.2. *If R is an UFD then there exists a valuation on R , then R is a Euclidean domain.*

Corollary 3.3. *Let R be a ring and P is a prime ideal of R . If R has a P -adic filtration and $R = \bigcup_{i=0}^{+\infty} P^i$, then R is a Euclidean domain.*

Proof. By proposition (3.2) R is a discrete valuation and so by proposition (2.1) R is a Euclidean domain.

□

Corollary 3.4. *Let R be a PID then R is a Euclidean domain.*

Proof. By proposition (3.3) and proposition (2.1) we have R is a Euclidean domain. \square

Corollary 3.5. *Let R be a UFD then R is a Euclidean domain.*

Corollary 3.6. *Let R be a strongly filtered ring. Then R is Manis ring.*

Corollary 3.7. *Let P is a prime ideal of R and $\{P^n\}_{n \geq 0}$ be P -adic filtration. Then R is Manis ring*

Proposition 3.4. *Let R_ν be a Manis ring. If R_ν is ν -closed, then R_ν is Prüfer.*

Proof. See proposition 1 of [9] \square

Proposition 3.5. *Let R be a strongly filtered ring. Then R is ν -closed.*

Proof. By proposition (3.1) and definition (2.9) we have R is Manis ring and $R = R_\nu$.

Now let $\alpha, \beta \in R$ and

$$\nu(\alpha) = i \text{ and } \nu(\beta) = j$$

Consequently if

$$(\alpha + \nu^{-1}(\infty))(\beta + \nu^{-1}(\infty)) \in \nu^{-1}(\infty)$$

Then $i + j \geq \infty$ so $\alpha \in \nu^{-1}(\infty)$ or $\beta \in \nu^{-1}(\infty)$. Hence by definition (2.11) R is ν -closed. \square

Corollary 3.8. *Let R be a strongly filtered ring. Then R is Prüfer.*

Proof. By proposition (3.6) R is ν -closed so by proposition (3.4) R is Prüfer. \square

Proposition 3.6. *Let P is a prime ideal of R and $\{P^n\}_{n \geq 0}$ be P -adic filtration. Then R is ν -closed.*

Proof. By proposition (3.2) and definition (2.9) we have R is Manis ring and $R = R_\nu$.

Now let $\alpha, \beta \in R$ and

$$\nu(\alpha) = i \text{ and } \nu(\beta) = j$$

Consequently if

$$(\alpha + \nu^{-1}(\infty))(\beta + \nu^{-1}(\infty)) \in \nu^{-1}(\infty)$$

Then $i + j \geq \infty$ so $\alpha \in \nu^{-1}(\infty)$ or $\beta \in \nu^{-1}(\infty)$. Hence by definition (2.11) R is ν -closed. \square

Corollary 3.9. *Let P is a prime ideal of R and $\{P^n\}_{n \geq 0}$ be P -adic filtration. Then R is Prüfer.*

Proof. By proposition (3.6) R is ν -closed so by proposition (3.4) R is Prüfer. \square

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